Noise Fundamentals

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Abstract

Johnson noise *and* ***Shot noise*** *are two major sources of electrical noise. Each of them is dependent upon a physical constant. Johnson noise corresponds with Boltzmann's constant (k_b) which can be measured with Nyquist's theorem. Shot noise corresponds to the charge of an electron, which can be measured with Schottkey's theorem. Unfortunately, there are multiple other sources of noise that need to be accounted for, and the amplitude of the noise is very small and spurious, so it needs to be largely amplified and manipulated. We measured the charge of an electron as 1.61e-19* $\pm 2.55e-21$ *Coulombs with a variation of 3.53e-40. The actual value for the charge of the electron is 1.602e-19 Coulombs. For the Boltzmann's constant, we measured 1.36e-23*$m^2kgs^{-2}K^{-1} \pm 5.52e-26$ *with a variation of 1.28e-49. The actual value is 1.38e-23*$m^2kgs^{-2}K^{-1}$.

I Introduction

Noise is present across measurements that must be made in physics as distortions to a signal. For example, seismometer noise is the signal that a seismometer picks up without any seismic activity. Electrical noise is present when measuring electric signals. Two of the sources of noise intrinsic to electronics are Johnson noise and shot noise.

I.I Johnson Noise

Johnson noise is the noise from thermodynamic fluctuations. Electrons bouncing off of each-other randomly produces small currents. Since electron movement is random, the average noise is 0, but at any given time there is a small positive or negative voltage across in, for example, a resistor.

Nyquist's original discussion on black body emissions best describe the phenomenon of Johnson noise. Illustrated below (Figure 1) is a source resistor taken to be ideal connected by a transmission line L to a voltage source.

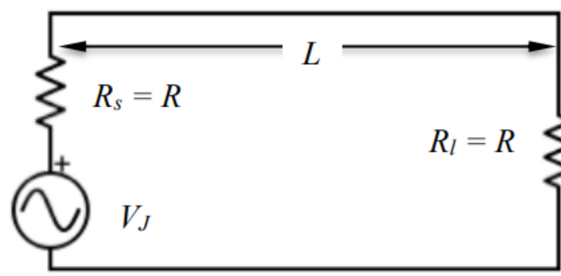

Figure 1: An ideal resistor connected to a voltage supply across a line of length L [1]:Lab Handout

Propagating electric modes, moving at v = c have the boundary condition that the potential at the ends of the line be equal. This is satisfied when $n\lambda = L$. Frequency relates to λ by $c = f\lambda$. Thus the number of propagating modes relative to frequency is L/C. Below is the energy dependant upon temperature given by the Planck distribution.

$$\frac{hf}{(e^{hf}/k_bt - 1)} \quad (1)$$

With this information, we can find the average power emitted from R_s as

$$P_e = R_s < I_j^2 >= R_s * < (\frac{V_j}{R_s + R_l})^2 >$$

This reduces to

$$P_e =< V_j^2(t) > /R$$

Using the Planck distribution we can form the Nyquist theorem and relate k_b to measurable variables. If we can measure the Johnson voltage across a resistor R at a temperature T, we can calculate Boltzmann's constant. We can use high and low pass filters to set a known ΔF

$$< V_j^2(t) >= 4k_bTR\Delta F \quad (2)$$

I.II Shot Noise

While Johnson noise comes from thermal fluctuations. Shot noise comes from charge quantization and can be observed in currents. In a flow of electrons from cathode to anode, the number of electrons hitting the anode, which is equivalent to the current is proportional to the shot noise.

$$I_s^2 = \frac{1}{N} \sum (I_i - < I >)^2$$

Current is equal to the flow of electrons so

$$I_s^2 = (\frac{e}{\Delta t})^2 \sum (n_i - < n >)^2 \tag{3}$$

The rms value of n is $\sqrt{n}$, the shot noise is

$$I_s^2 = 2e < I > \Delta F = 2eI_{dc}\Delta F$$

The result is Schottkys theorem relating the charge of an electron to the DC current, its frequency range and the shot noise it generates.

Both shot and Johnson noise are characterized as white noise, they exhibit flat distributions across a frequency bandwidth. By using Schottky's theorem, we can measure the charge of an electron by isolating the shot noise from a photo current with known parameters and an isolated bandwidth. Our goal is to measure Boltzmann's constant and the charge of an electron.

II Experimental Procedure

II.I Johnson Noise Resistance Dependence

There are two main sections to this experiment: studying Shot noise and studying Johnson noise. Because the Johnson and Shot noise signals are too small to accurately be picked up by a voltmeter, we need to process them with an apparatus first. To measure these noises we connect a resistor to an amplifier and then to a high and low pass filter to limit the frequencies we're measuring in noise. Recall equations (2) and (3) both depend on ΔF. We set the cutoff frequencies to .1kHz and 100kHz. Next we need to amplify the signal some more (we used a gain of 600 for the initial amplifier then passed it through another amplifier with a gain of 1500 after the filters, the total gain is then 900,000). When this amplified signal is displayed on an oscilliscope, it fluctuates around 0 as expected. To get a meaningful measure of the noise we square the signal with a multiplier op amp. Now we can read out the mean square noise. We need to square the noise after amplifying it because as mentioned in I.I, the noise fluctuates around 0 and the mean value would come out to 0. Shottky and Nyquist's theorems depend on the squared noise because of this.

The output voltage includes the mean squared noise from other noise sources however, and we need to factor that out to solve for $< V_j^2 >$. The output voltage when the gain of the preceding circuits is accounted for is:

$$V_{out} = \frac{(< V_j^2 > + < V_n^2 >) * (G_1 G_2)^2}{10} \tag{4}$$

When R is low, V_j is low (recall Nyquist's theorem). For R = 1 Ω for example, the V_n term dominates. Thus the V_n term be measured by setting the source resistance to 1 Ω, this will drop the V_j term in equation 4. We can then hold it constant as it doesn't change with R in future calculations of V_j. When R = 1 Ω:

$$V_{out} = \frac{(< V_n^2 >) * (G_1 G_2)^2}{10}$$

To measure Boltzmann's constant, we can plot the power spectral density given by $S =< V_j^2 > /\Delta F$ vs. the resistance. Recall from (2) that at a fixed temperature we can find k_b. By holding the temperature constant and collecting the output voltage with varying feedback resistors we can measure k_b. ΔF is close to the area under a white noise curve between the two cutoff frequencies (i.e. the difference between the two cutoff frequencies). However, because the high and low pass filters are imperfect, they follow a Gaussian distribution pattern more closely. More on this will be discussed in the next section, for this part of the experiment we can gather ΔF from the following table:

		f_{LP}					
		0.33 kHz	1.0 kHz	3.3 kHz	10 kHz	33 kHz	100 kHz
f_{HP}	10 Hz	355	1,100	3,654	11,096	36,643	111,061
	30 Hz	333	1,077	3,632	11,074	36,620	111,039
	100 Hz	258	1,000	3,554	10,996	36,543	110,961
	300 Hz	105	784	3,332	10,774	36,321	110,739
	1000 Hz	9.0	278	2,576	9,997	35,543	109,961
	3000 Hz	0.4	28	1,051	7,839	33,324	107,740

Figure 2: Notice the ΔF is very close to the difference between the two cut off frequencies. [1]:Lab Handout

II.II Johnson Noise Temperature Dependence

Since Johnson noise also depends on T, we can hold R constant and change the temperature to measure Boltzmann's constant. At T = 0K, there should be no Johnson noise present. We attach our amplifier to an external feedback resistor that is submerged in a vacuum flask containing liquid nitrogen to cool it down. The apparatus in this section is the same circuit and processing used in the resistance dependence section, however the resistor is submerged in liquid nitrogen in a dewar. Since we cannot stick a thermometer into a dewar full of liquid nitrogen, there is a diode next to the resistor. The voltage across the diode can be used to measure temperature, below is a calibration curve correlating the voltage of the diode with the temperature within the dewar (temperature of R). (Figure 3)

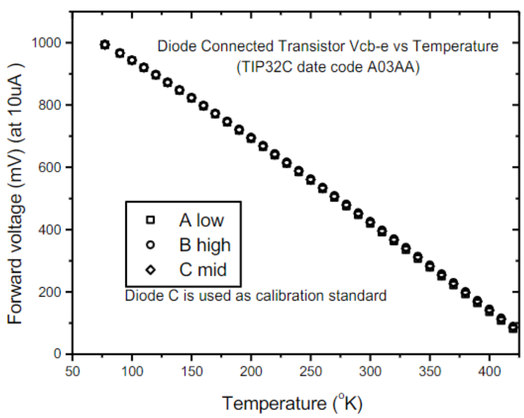

Figure 3: Diode calibration to find the temperature of the external feedback resistor [1]:Lab Handout

Now if we find the Johnson noise using the same method as in the resistance dependence section, we can find k_b through its temperature dependence. However, there is one more factor that must be accounted for. Because of the capacitive coupling of the wire connecting the external resistor to the electronics, the frequency bandwidth changes. We can no longer use the values from Figure 2 because of this.

The filters in the electronics box can be modeled with the Buttersworth filter response [2]

$$< V_j^2 >= \int_{f_1}^{f^2} S df = \int_0^\infty S(G_{LP}G_{HP})^2 df$$

Relating this to the definition of noise power spectral density gives us

$$\Delta f = \int_0^\infty (G_{LP}G_{HP})^2 df$$

This yields the results to the table of figure 2 in Mathematica. There wire adds a capacitive component that can attenuate high frequency components of the noise. The transmission function of the diode probe is:

$$G_c = \frac{1}{\sqrt{1 + (f/f_c)^2}}$$

Here, f_c is given by

$$f_c = \frac{1}{2 * \pi * RC}$$

We can add this to our equation for ΔF and get:

$$\Delta f = \int_0^\infty (G_{LP}G_{HP} * G_C)^2 df \qquad (5)$$

We solved this integral for a value of 9957.4 where our high pass cutoff voltage is 1kHz, our low pass cutoff voltage is 10kHz and the capacitance of the wire is estimated at 100pF.

Now we have the full relationship between the calibration voltage and the Johnson noise. If we know the calibration voltage, we can find the temperature. We can also calculate ΔF using equation 5. Lastly, we can calculate the non-Johnson noise with the same method as last time, see equation 4. With the relationship of temperature to Johnson noise, we can once again calculate Boltzmann's constant.

II.III Shot Noise

Using an illuminated photodiode we can create a stream of electrons. To measure shot noise, we switch from a simple amplifier measuring the noise in a resistor to a reverse biased photodiode. The circuit below shows how the current passing through the photodiode passes through R_f:

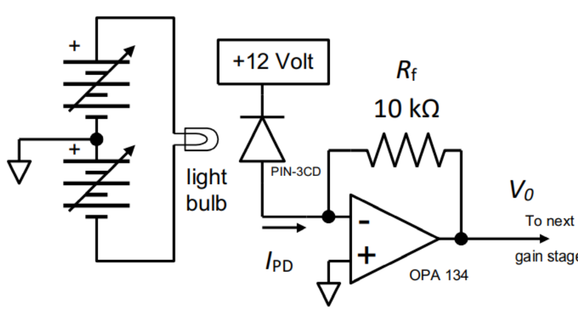

Figure 4: A transimpedance amplifier used to preamplify a stream of electrons [1]:Lab Handout

The output signal (V_0 is then proportional to the photocurrent, which can be adjusted by changing the brightness of the light. The output noise from this configuration after running it through the electronics is:

$$V_{out} = \frac{(R_f^2 * I_{shot}^2 + < V_n^2 >)(100 * G_2)}{10V}$$

V_n can be calculated in a similar fashion to what we did in the Johnson noise section to calculate the non Johnson noise. Instead of eliminating the Johnson noise factor, we turn off the light so I_{shot} will be 0. Then we have the V_n, which will remain constant at varying brightness. V_n in this case is:

$$V_n = \frac{V_{out} * 10}{((100 * G_2) * *2)}$$

Finally, we collect data at varying the photocurrent. The DC current through the light relates to the shot current by Schottky's theorem. This is equation 3 found in section I.I. After varying the photocurrent and holding the bandwidth constant, we vary the bandwidth holding the photocurrent constant so we have the shot noise relationship with both bandwidth and the lightbulbs current. Now we can calculate the charge of an electron with each of these relationships. Recall that the bandwidth can be found using the table in Figure 2 again because we no longer have the capacitive component affecting the ΔF.

III Results and Analysis

III.I Johnson Noise Results

Firstly, we measured the noise at 1 Ω, since the RMS voltage varies we measured this 20 times and took the mean for our V_{out} to calculate V_n

We get .5732 as our V_{out} and 0.00022 as the standard error. Plugging into the equation from the previous section:

$$V_n = \frac{V_{out} * 10}{((G_1 * G_2) * *2)} = \frac{.5732 * 10}{((600 * 1500) * *2)}$$

$$= 7.08e - 12$$

After collecting the output voltage using four resistors: $10\Omega, 100\Omega, 1k\Omega, 10k\Omega$ with 10 data points for each to capture the volitility of the output voltage, we can then calculate the Johnson noise for each datapoint. With this we found the spectral density and thus k_b for each data point. The data can be referenced in the references section.

Recall:

$$V_j = \frac{V_{out} * 10}{((G_1 * G_2) * *2)} - V_n$$

$$k_b = \frac{< V_n^2 > *10}{4\Delta F}$$

We got an average k_b of 1.53e-23 with a variation of 5.88e-25. This high variation and overestimate is mostly due to the data collected with the 10Ω data, with its average being a whopping 1.85e-23. Omitting this data would yield a much closer value to Boltzmanns constant and it includes the majority of the outliers. More on why I think this is in the conclusion.

Plotting power spectral density vs R shows k_b which goes with the slope. When plotting the power spectral density vs. R we get figure 5.

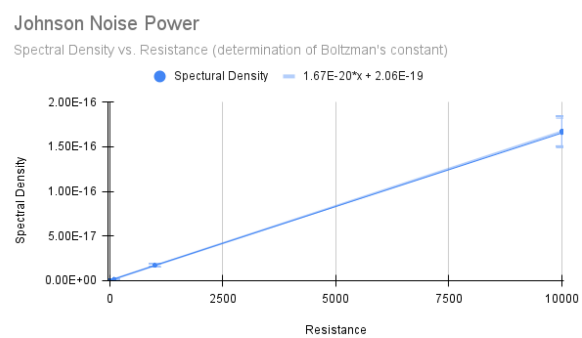

Figure 5: Power spectral density as a function of resistance [3]:Data

Next, for the temperature dependence we recorded the calibration voltage vs the output voltage. Calibration voltage can be used to calculate temperature (see II.II). We also solved the integral for (see II.II) to yield 9957.4Hz. We can now calculate k_b using the method we used to it in the resistance section. With the temperature dependent measurements at varying temperatures between 80-250K we get a Boltzmann's constant of 1.36e-23$\pm 5.52e-26$. This is a much lower variation and a much closer approximation to Boltzmanns constant! The actual value from Nyquist's theorem is 1.38e-23. Plotting temperature against power spectral density, again where Boltzmann's constant goes with the slope, we get figure 6.

III.II Shot Noise Results

For the first part, holding the lightbulbs current fixed and varying the bandwidth range we calculated the average charge of an electron from our data to be 1.90e-19. Our variation here was 1.77e-20. Quite large, however, the variation within each band was smaller so this may be due to ΔF variation. Recall the internal high pass filter is 16Hz and the low pass is varied. We did not need to consider the capacitive coupling of the wire connecting

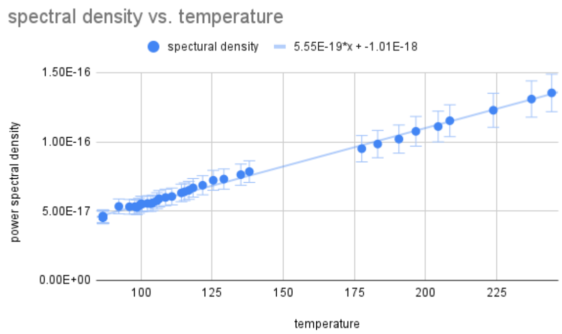

Figure 6: Power spectral density as a function of temperature [3]:Data

to the dewar because that circuit is external to the shot noise circuit.

Plotting the power spectral density vs I_{dc} we get a linear relationship. (Figure 7)

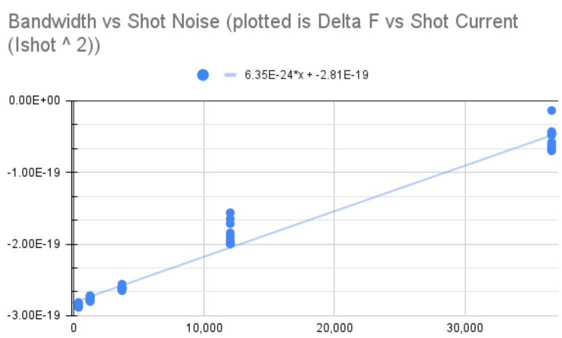

Figure 7: Bandwidth as a function of $< I^2_{shot}$. [3]:Data

This makes sense. More noise will pass through the filters if the ΔF is larger. This plot shows that $< I^2_{shot} >$ goes with ΔF

III.III Shot Noise Photocurrent Dependence

Next, we vary I_{dc} to calculate the charge of an electron. We hold ΔF constant. In our case $\Delta F = 111650$.

We can once again calculate the component of noise not attributed to shot noise by turning off the light. This is:

$$< V_n^2 > \frac{V_{out} * 10}{(100 * G2) * *2}$$

From our data on V_{dc} and V_{out} we calculated a column of I_{shot}, I_{dc}, S and e, the charge of an electron.

Recalling:

$$< I^2_{shot} >= \frac{(10 * V_{out}/((100 * G2) * *2) - < V_n^2 >)}{(Rf * *2)}$$

$$e = \frac{< I^2_{shot} >}{2 I_{dc} \Delta F}$$

With this we get the charge of the electron mean as: 1.61e-19 with a variation of 2.63e-21. This is much better than our bandwidth varying measurement.

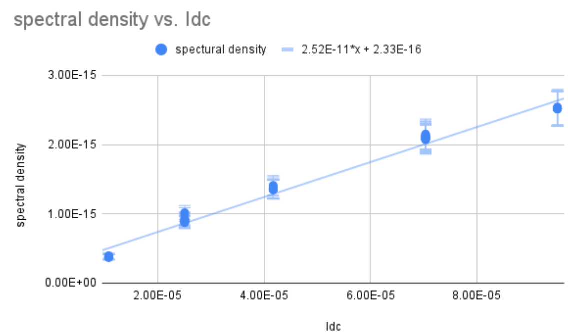

Figure 8: Power spectral density as a function of current through the photo diode [3]:Data

This slope goes with the charge of an electron given the mathematics shown in II.III.

IV Conclusion

In conclusion, we were able to find the charge of an electron and Boltzmann's constant with a small uncertainty. For the charge of an electron, we will take our second value with the lower variation, 1.61e-19$\pm 2.55e - 21$ Coulombs. The actual value for the charge of the electron is 1.602e-19 Coulombs. For the Boltzmann's constant using the temperature dependent data we get 1.36e-23$m^2 kg s^{-2} K^{-1}$ with a variation of 5.51e-26. The actual value is 1.38e-23$m^2 kg s^{-2} K^{-1}$. I believe the resistance data is less reliable for a few reasons. Firstly, the 10Ω resistor consistently returned higher values for noise than it should've whereas the other resistors did not. I believe this is because it likely has a slightly higher resistance than the 10Ω advertised. Even a 1Ω increase could explain the unexpectedly high noise for this resistor. Unfortunately, its internal to the machinery so we didn't want to take it apart to measure it. It could also be that a higher proportion of the noise was non-Johnson noise, and since we were measuring a very small amount of noise (since the resistor is relatively small) we may have overestimated. V_n

Our power spectral density relation graphs match our expectations, if we had more time we would liked to have investigated the power spectral density relations to other variables to confirm that they match the current literature.

Finally, to quickly compute entire columns of data, I created a simple python program. It is cited below. The lab manual and data are also referenced below. This is where variation was calculated.

References

[1] D. W. Beyermann, *Noise Fundementals*. UC Riverside, 2022.

[2] P. Horowitz and W. Hill in *The Art of Electronics, 3rd edition* (C. University, ed.), ch. 4, 2016.

[3] S. Field, "Data and python scripts," *https://github.com/sevdeawesome/Noise-Fundementals*, 2022.